PROPOSED
MARKING GUIDE
PURE MATHEMATICS
P425/1 2023

NO	SOLUTION	MKS	COMMENT
1	$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(2n+1)(n+1)$ **Solution** For $n = 1$; L.H.S $= 1^2 = 1$, R.H.S $= \frac{1}{6} \times 1 \times (3)(2) = 1$ It holds For $n = 2$; L.H.S $= 1^2 + 2^2 = 5$ R.H.S $= \frac{1}{6} \times 2 \times (5)(3) = 5$ It holds *Assume the result holds for $n = k$* $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(2k+1)(k+1)$ For $n = k+1$; $1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{1}{6}k(2k+1)(k+1) + (k+1)^2$ R.H.S $= \frac{1}{6}k(2k+1)(k+1) + (k+1)^2$ $= \frac{k+1}{6}[2k^2 + k + 6k + 6]$ $= \frac{k+1}{6}[2k^2 + 7k + 6]$ $= \frac{1}{6}(k+1)(2k+3)(k+2)$ It holds for $n = k+1$		
		05	
2	If $y = mx + c$ is a tangent to $4x^2 + 3y^2 = 12$, then $4x^2 + 3(mx+c)^2 = 12$		

	$4x^2 + 3(m^2x^2 + 2mcx + c^2) = 12$ $4x^2 + 3m^2x^2 + 6mcx + 3c^2 = 12$ $(4 + 3m^2) + 6mcx + 3c^2 - 12 = 0$ For tangency, $b^2 = 4ac$ $(6mc)^2 = 4(4 + 3m^2)(3c^2 - 12)$ $36m^2c^2 = 4(12c^2 - 48 + 9m^2c^2 - 36m^2)$ $9m^2c^2 = 12c^2 - 48 + 9m^2c^2 - 36m^2$ $12c^2 = 48 + 36m^2$ $\therefore c^2 = 4 + 3m^2$		
		05	
3	$y = e^x \cos 3x$ $\frac{dy}{dx} = -3e^x \sin 3x + e^x \cos 3x$ $\frac{dy}{dx} = -3e^x \sin 3x + y$ $\frac{d^2y}{dx^2} = -3[3e^x \cos 3x + e^x \sin 3x] + \frac{dy}{dx}$ $= -9y - 3e^x \sin 3x + \frac{dy}{dx}$ $= -9y + \frac{dy}{dx} - y + \frac{dy}{dx}$ $= 2\frac{dy}{dx} - 10y$ $\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$		
		05	
4	Let $\boldsymbol{d} = 3\boldsymbol{i} + 12\boldsymbol{j} + 4\boldsymbol{k}$ and $n = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ Let $\theta =$ required angle Using $\boldsymbol{d} \cdot \boldsymbol{n} = \lvert\boldsymbol{d}\rvert\lvert\boldsymbol{n}\rvert \sin\theta$ $\begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \sqrt{3^2 + 12^2 + 4^2}\sqrt{(-1)^2 + 2^2 + 2^2}\ \sin\theta$ $-3 + 24 + 8 = \sqrt{169}\sqrt{9}\ \sin\theta$		

$29 = 13 \times 3 \sin\theta$

$\sin\theta = \frac{29}{39}$

$\theta = \sin^{-1}\left(\frac{29}{39}\right)$

$\theta = 48.04^0$

05

5

$\frac{7-2x}{(x+1)(x-2)} > 0$

Critical values

$x = -1, x = 2, x = \frac{7}{2}$

x	$x < -1$	$-1 < x < 2$	$2 < x < 3.5$	$x > 3.5$
$(7-2x)$	$+$	$+$	$+$	$-$
$(x+1)(x-2)$	$+$	$-$	$+$	$+$
$\frac{7-2x}{(x+1)(x-2)}$	$+$	$-$	$+$	$-$

$\therefore$ The range of values of x are: $x < -1, 2 < x < 3.5$

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6

$\int_0^{\pi/3}(1+\cos 3y)^2\,dy = \int_0^{\pi/3}(1 + 2\cos 3y + cos^2 3y)\,dy$

$= \int_0^{\pi/3}\left[1 + 2\cos 3y + \frac{1}{2}(\cos 6y + 1)\right]dy$

$= \left[y + \frac{2}{3}\sin 3y + \frac{1}{12}\sin 6y + \frac{1}{2}y\right]_0^{\pi/3}$

$= \left[\frac{3}{2}y + \frac{2}{3}\sin 3y + \frac{1}{12}\sin 6y\right]_0^{\pi/3}$

$= \left(\frac{\pi}{2} + \frac{2}{3}\sin\pi + \frac{1}{12}\sin 2\pi\right) - 0$

$= \frac{\pi}{2}$ or 1.5708

05

7

Let $2\sin\theta + 3\cos\theta \equiv R\sin(\theta+\alpha)$

$2\sin\theta + 3\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$\equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

Comparing coefficients of;

$\sin\theta$; $R\cos\alpha = 2$ ………………….(i)

	$\cos\theta$; $R\sin\alpha = 3$ …………………(ii) $(R\cos\alpha)^2 + (R\sin\alpha)^2 = 2^2 + 3^2$ $R^2(cos^2\alpha + sin^2\alpha) = 4 + 9 = 13$ $R^2 = 13$ $R = \sqrt{13}$ (ii)÷(i); $\tan\alpha = \frac{3}{2}$ $\alpha = \tan^{-1}(1.5)$ $\alpha = 56.31^0$ $\therefore 2\sin\theta + 3\cos\theta = \sqrt{13}\sin(\theta + 56.31^0)$		
		05	
8	Let $f(x) = \ln(2+x), f(0) = \ln 2$ $f'(x) = \frac{1}{2+x}, f'(0) = \frac{1}{2}$ $f''(x) = -(2+x)^{-2}\cdot 1 = \frac{-1}{(2+x)^2}, f''(0) = -\frac{1}{4}$ Using $f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$ $\therefore \ln(2+x) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$		
		05	
9	a) Let $f(z) = z^3 - 7z^2 + 19z - 13$ Putting $z = 1$ $f(1) = 1^3 - 7(1)^2 + 19(1) - 13$ $f(1) = 0$ $z = 1$ is a root and then $z - 1$ is a factor		

$$\begin{array}{r|l} & z^2-6z+13 \\ (z-1) & z^3-7z^2+19z-13 \\ & -\underline{(z^3-z^2)} \\ & \quad -6z^2+19z-13 \\ & \quad -\underline{(-6z^2+6z)} \\ & \qquad 13z-13 \\ & \qquad -\underline{(13z-13)} \\ & \qquad - \quad - \end{array}$$

$z^2-6z+13=0$

$z=\frac{6\pm\sqrt{(-6)^2-4\times1\times13}}{2\times1}$

$z=\frac{6\pm\sqrt{-16}}{2}$

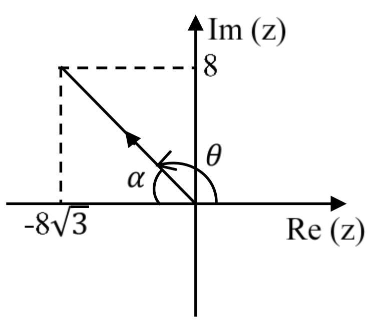
$z=\frac{6\pm4i}{2}=3\pm2i$

The values of z are $1, 3+2i$ and $3-2i$

b) $8(-\sqrt{3}+i)=-8\sqrt{3}+8i$

Let $z=-8\sqrt{3}+8i$

$r=|z|=\sqrt{(-8\sqrt{3})^2+8^2}=16$ units



arg(z) $=\theta=180^0-\tan^{-1}\left(\frac{8}{8\sqrt{3}}\right)=180^0-30^0=150^0=\frac{5\pi}{6}$

Using $z=r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$

$z=16^{\frac{1}{4}}\left[\cos\left(\frac{\frac{5\pi}{6}+2\pi k}{4}\right)+i\sin\left(\frac{\frac{5\pi}{6}+2\pi k}{4}\right)\right]$

	$z = 2\left[\cos\left(\frac{5\pi+12\pi k}{24}\right) + i\sin\left(\frac{5\pi+12\pi k}{24}\right)\right]$ For $k = 0, z_1 = 2\left[\cos\left(\frac{5\pi}{24}\right) + i\sin\left(\frac{5\pi}{24}\right)\right]$ $= 2(0.7934 + 0.6088i)$ $= 1.5868 + 1.2176i$ For $k = 1, z_2 = 2\left[\cos\left(\frac{17\pi}{24}\right) + i\sin\left(\frac{17\pi}{24}\right)\right]$ $= 2(-0.6088 + 0.7934)$ $= -1.2176 + 1.5868i$ For $k = 2, z_3 = 2\left[\cos\left(\frac{29\pi}{24}\right) + i\sin\left(\frac{29\pi}{24}\right)\right]$ $= 2(-0.7934 - 0.6088i)$ $= -1.5868 - 1.2176i$ For $k = 3, z_4 = 2\left[\cos\left(\frac{41\pi}{24}\right) + i\sin\left(\frac{41\pi}{24}\right)\right]$ $= 2(0.6088 - 07934i)$ $= 1.2176 - 1.5868i$		
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10	**Method I** Let $\frac{3x^3+2x^2-3x+1}{x(1-x)} \equiv Ax + B + \frac{C}{x} + \frac{D}{1-x}$ $3x^3 + 2x^2 - 3x + 1 \equiv x(Ax + B)(1 - x) + C(1 - x) + Dx$ Putting $x = 1; 3 = D$ $\therefore D = 3$ Putting $x = 0; 1 = C$ $\therefore C = 1$ Comparing coefficients of; $x^3; 3 = -A$ $\therefore A = -3$ $x^2; 2 = A - B$ $2 = -3 - B$ $\therefore B = -5$ $\therefore \frac{3x^3+2x^2-3x+1}{x(1-x)} \equiv -3x - 5 + \frac{1}{x} + \frac{3}{1-x}$ Hence;		

$$\int f(x)\,dx = \int(-3x-5)\,dx + \int \frac{1}{x}dx + \int \frac{3}{1-x}dx$$

$$= -\frac{3}{2}x^2 - 5x + \ln x - 3\ln(1-x) + c$$

Method II

$$\frac{3x^3+2x^2-3x+1}{x(1-x)} = \frac{3x^3+2x^2-3x+1}{x-x^2}$$

$$\begin{array}{r|l} & \quad -3x-5 \\ \hline (-x^2+x) & \; 3x^3+2x^2-3x+1 \\ & -\;3x^3-3x^2 \\ \hline & \qquad\; 5x^2-3x+1 \\ & \qquad -\;5x^2-5x \\ \hline & \qquad\qquad\quad 2x+1 \end{array}$$

$$\frac{3x^3+2x^2-3x+1}{x(1-x)} = -3x-5+\frac{2x+1}{x(1-x)}$$

Let $\frac{2x+1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x}$

$2x+1 \equiv A(1-x) + Bx$

Putting $x = 1; 3 = B$ $\quad\therefore B = 3$

Putting $x = 0; 1 = A$ $\quad\therefore A = 1$

$\therefore \frac{3x^3+2x^2-3x+1}{x(1-x)} \equiv -3x-5+\frac{1}{x}+\frac{3}{1-x}$

Hence;

$$\int f(x)\,dx = \int(-3x-5)\,dx + \int \frac{1}{x}dx + \int \frac{3}{1-x}dx$$

$$= -\frac{3}{2}x^2 - 5x + \ln x - 3\ln(1-x) + c$$

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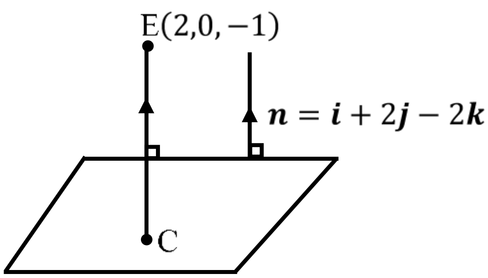
11 a) Equation of a line through $\mathrm{E}(2,0,-1)$

$$\boldsymbol{r} = \begin{pmatrix}2\\0\\-1\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

Let $\boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

	$x = 2 - 2\mu$ $y = \mu$ $z = -1 + 2\mu$ At point B; $2 - 2\mu + 2\mu - 2(-1 + 2\mu) = 8$ $2 + 2 - 4\mu = 8$ $-4\mu = 4 \qquad \therefore \mu = -1$ ⇨ $x = 2 - 2(-1) = 4$ $y = -1$ $z = -1 + 2(-1) = -3$ $\therefore B(4, -1, -3)$ b)  Equation of the perpendicular from E to the plane; $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ Let $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ $x = 2 + t$ $y = 2t$ $z = -1 - 2t$ At point C;		

	$2 + t + 4t - 2(-1 - 2t) = 8$ $2 + 5t + 2 + 4t = 8$ $9t = 4$ $t = \frac{4}{9}$ $\Rightarrow\ x = 2 + \frac{4}{9} = \frac{22}{9}$ $y = 2\left(\frac{4}{9}\right) = \frac{8}{9}$ $z = -1 - 2\left(\frac{4}{9}\right) = -\frac{17}{9}$ $\therefore C\left(\frac{22}{9}, \frac{8}{9}, -\frac{17}{9}\right)$		
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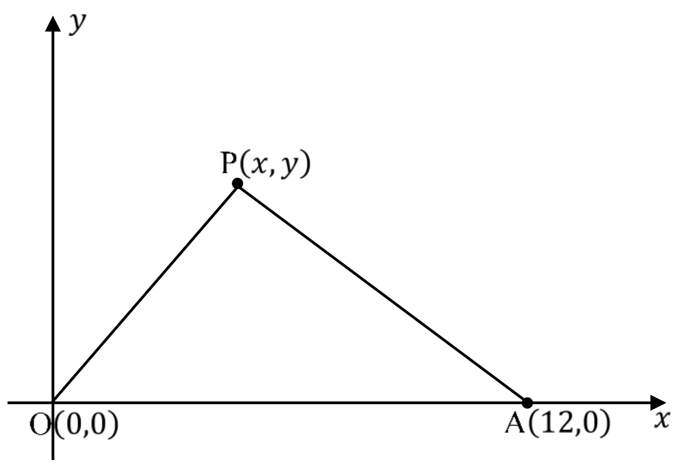
12	*a)* *No. of ways* $= 10! = 3{,}628{,}800$ *ways* *b)* *No. of ways* $= {}^{9}C_{6} \times {}^{7}C_{5} = 84 \times 21 = 1764$ *ways* *c)* ${}^{20}C_{r} = {}^{20}C_{r-2}$ $\frac{20!}{(20-r)!r!} = \frac{20!}{(20-(r-2))!(r-2)!}$ $(20 - r)!\,r! = \big(20 - (r - 2)\big)!\,(r - 2)!$ $(20 - r)!\,r! = (22 - r)!\,(r - 2)!$ $(20 - r)!\,r(r - 1)(r - 2)! = (22 - r)(21 - r)(20 - r)!\,(r - 2)!$ $r(r - 1) = (22 - r)(21 - r)$ $r^2 - r = 462 - 22r - 21r + r^2$ $-r = 462 - 43r$ $42r = 462$ $r = 11$ ***Alternatively:*** If ${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x + y = n$ Then ${}^{20}C_{r} = {}^{20}C_{r-2}$ $r + r - 2 = 20$ $2r = 22$		

	$\therefore r = 11$		
		12	
13	a) $x = t^2 - 3, y = t(t^2 - 3)$ From $x = t^2 - 3$ $t = \sqrt{x+3}$ ⇨ $y = \sqrt{x+3}(x)$ Squaring both sides gives $y^2 = x^2(x+3)$ $\therefore y^2 = x^3 + 3x^2$ or $x^3 = y^2 - 3x^2$ ***Alternatively:*** $y = tx \Rightarrow t = \frac{y}{x}$ Using $x = t^2 - 3$ ⇨ $x = \frac{y^2}{x^2} - 3$ $\therefore x^3 = y^2 - 3x^2$ or $y^2 = x^3 + 3x^2$ b) (i)  $\overline{OP} = 5\overline{PA}$ $\overline{OP}^2 = 25\overline{PA}^2$ $(x-0)^2 + (y-0)^2 = 25[(x-12)^2 + (y-0)^2]$ $x^2 + y^2 = 25(x^2 - 24x + 144 + y^2)$		

	$x^2 + y^2 = 25x^2 + 25y^2 - 600x + 3600$ $24x^2 + 24y^2 - 600x + 3600 = 0$ $x^2 + y^2 - 25x + 150 = 0$ hence a circle (ii) Completing squares $x^2 + y^2 - 25x = -150$ $\left(x - \frac{25}{2}\right)^2 + (y-0)^2 = -150 + \left(\frac{25}{2}\right)^2$ $\left(x - \frac{25}{2}\right)^2 + (y-0)^2 = \frac{25}{4}$ $\therefore$ Centre, $C\left(\frac{25}{2}, 0\right)$ and radius, $r = \sqrt{\left(\frac{25}{4}\right)} = \frac{5}{2} = 2.5$ units		
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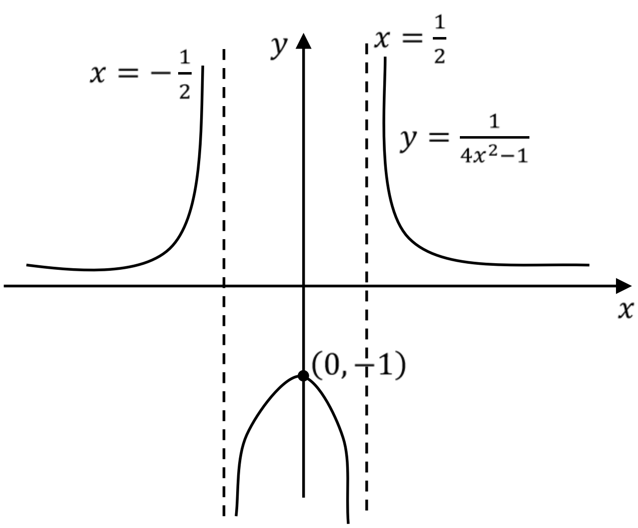
14	a) Turning points $\frac{dy}{dx} = \frac{(4x^2-1)\cdot 0 - 1\cdot 8x}{(4x^2-1)^2} = 0$ $8x = 0$ $x = 0$ When $x = 0, y = \frac{1}{0-1} = -1$ $\therefore (0, -1)$ Nature; $\frac{d^2y}{dx^2} = \frac{(4x^2-1)^2\cdot -8 + 8x\cdot 2(4x^2-1)\cdot 8x}{(4x^2-1)^4} = \frac{(4x^2-1)(96x+8)}{(4x^2-1)^4}$ When $x = 0, y = \frac{(0-1)(0+8)}{(0-1)^4} = -8 < 0$ $\therefore (0, -1)_{max}$ b) Asymptotes Vertical asymptote $4x^2 - 1 = 0$ $4x^2 = 1$ $x = \pm\frac{1}{2}$		

	$x=-\frac{1}{2}, x=\frac{1}{2}$ Horizontal asymptote $y=\frac{\frac{1}{x^2}}{4-\frac{1}{x^2}}$ As $x \to \pm\infty, y \to 0$ i.e $y=0$ Intercepts When $y=0, x=?$ $0=1, x$ is undefined When $x=0, y=?$ $y=\frac{1}{0-1}=-1, (0,-1)$  		
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15	a) $\tan 3\theta = \tan(2\theta+\theta)$ $=\frac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$ But $\tan 2\theta=\frac{2\tan\theta}{1-tan^2\theta}$ ⇨ $\tan 3\theta=\frac{\frac{2\tan\theta}{1-tan^2\theta}+\tan\theta}{1-\left(\frac{2\tan\theta}{1-tan^2\theta}\right)\cdot\tan\theta}$ $=\frac{2\tan\theta+\tan\theta-tan^3\theta}{1-tan^2\theta-2tan^2\theta}$		

$$= \frac{3\tan\theta - tan^3\theta}{1-3tan^2\theta}$$

$$\therefore \tan 3\theta = \frac{\tan\theta(3-tan^2\theta)}{(1-3tan^2\theta)}$$

ALT:

From De Movoire's theorem;

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$$= cos^3\theta + 3icos^2\theta\sin\theta - 3\cos\theta\, sin^2\theta - isin^3\theta$$

Equating components;

Real: $\cos 3\theta = cos^3\theta - 3\cos\theta\, sin^2\theta$ …………………(i)

Imaginary; $\sin 3\theta = 3cos^2\theta\sin\theta - sin^3\theta$……………(ii)

(ii)÷(i); $\tan 3\theta = \frac{3cos^2\theta\sin\theta - sin^3\theta}{cos^3\theta - 3\cos\theta sin^2\theta}$

Dividing through the R.H.S by $cos^3\theta$

$\tan 3\theta = \frac{3\tan\theta - tan^3\theta}{1-3tan^2\theta}$

$\therefore \tan 3\theta = \frac{\tan\theta(3-tan^2\theta)}{(1-3tan^2\theta)}$

b) $\cos 6x + \cos 2x + \cos 4x = 0$

$2\cos 4x\cos 2x + \cos 4x = 0$

$\cos 4x\,(2\cos 2x + 1) = 0$

Either $\cos 4x = 0$ or $2\cos 2x + 1 = 0$

For $\cos 4x = 0$

$4x = \cos^{-1}(0)$

$4x = 90^0, 270^0, 450^0, 630^0$

$x = 22.5^0, 67.5^0, 112.5^0, 157.5^0$

For $2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos^{-1}\left(\frac{1}{2}\right)$

$2x = 120^0, 240^0$

	$x = 60^0, 120^0$ $\therefore x = 22.5^0, 60^0, 67.5^0, 112.5^0, 120^0, 157.5^0$		
		12	
16	a) Let T be the body's temperature $\frac{dT}{dt} \propto (T-25)$ $\frac{dT}{dt} = -k(T-25)$ $\int \frac{dT}{T-25} = -\int k\, dt$ $\ln(T-25) = -kt + c$ $T - 25 = e^{-kt+c}$ $T - 25 = e^{-kt} \cdot e^{c}$ $T-25 = Ae^{-kt}$, $A = e^{c}$ $T = 25 + Ae^{-kt}$ When $t = 0, T = 90^0$C $90 = 25 + A$ $\therefore A = 65$ $T = 25 + 65e^{-kt}$ When $t = 6\ mins, T = 60^0$C $60 = 25 + 65e^{-6k}$ $e^{-6k} = \frac{35}{65}$ $-6k = \ln\left(\frac{35}{65}\right)$ $k = \frac{1}{6}\ln\left(\frac{65}{35}\right)$ $\therefore T = 25 + 65e^{-\frac{1}{6}\ln\left(\frac{65}{35}\right)\cdot t}$ b) When $T = 40^0, t = ?$ $40 = 25 + 65e^{-\frac{1}{6}\ln\left(\frac{65}{35}\right)\cdot t_1}$ $-\frac{1}{6}\ln\left(\frac{65}{35}\right) \cdot t_1 = \ln\left(\frac{15}{65}\right)$		

$$t_1 = \frac{-6\ln\left(\frac{15}{65}\right)}{\ln\left(\frac{65}{35}\right)} = 14.2124 \text{ minutes}$$

When $\mathrm{T} = 30^0, t = ?$

$$30 = 25 + 65e^{-\frac{1}{6}\ln\left(\frac{65}{35}\right)\cdot t_2}$$

$$-\frac{1}{6}\ln\left(\frac{65}{35}\right)\cdot t_2 = \ln\left(\frac{5}{65}\right)$$

$$t_2 = \frac{-6\ln\left(\frac{5}{65}\right)}{\ln\left(\frac{65}{35}\right)} = 24.8606 \text{ minutes}$$

$\therefore$ Time taken $= 24.8606 - 14.2124$

$= 10.6482 \approx 11$ minutes

Alternatively:

Using $\ln(T - 25) = -kt + c$

Set $t = t_1$ at $T = 40$ => $\ln 15 = -kt_1 + c$

Set $t = t_2$ at $\mathrm{T} = 30$ => $\ln 5 = -kt_2 + c$

Subtracting : $\ln 15 - \ln 5 = k(t_2 - t_1)$

The required time, $t_2 - t_1 = \frac{\ln 3}{k}$

$$= \ln 3 \div \frac{1}{6}\ln\left(\frac{65}{35}\right)$$

$$= \frac{6\ln 3}{\ln\left(\frac{65}{35}\right)}$$

$$= 10.6482 \approx 11 \text{ minutes}$$

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